Growth Model

Exogenous Growth Models: Malthusian and Solow

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Output-side real GDP at current PPPs

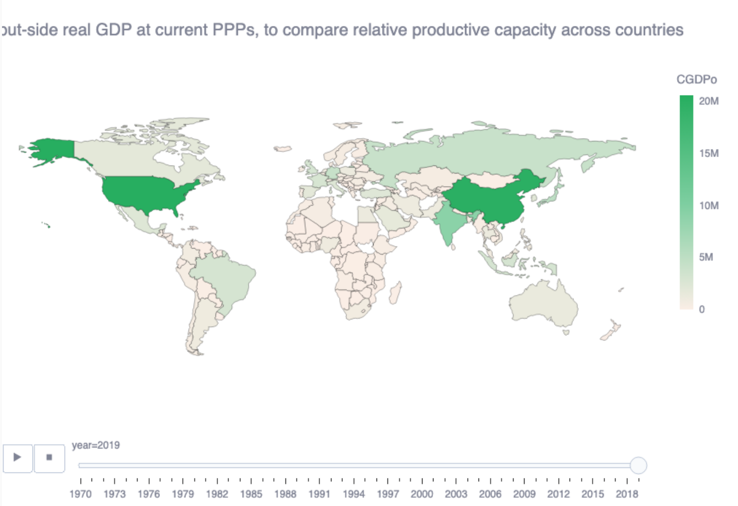


Figure 1: Source: Penn Table 10.1
https://www.rug.nl/ggdc/productivity/pwt/?lang=en

Introduction: Growth

Growth of output is one of the most talked number in macroeconomics. Some facts about growth within and across countries

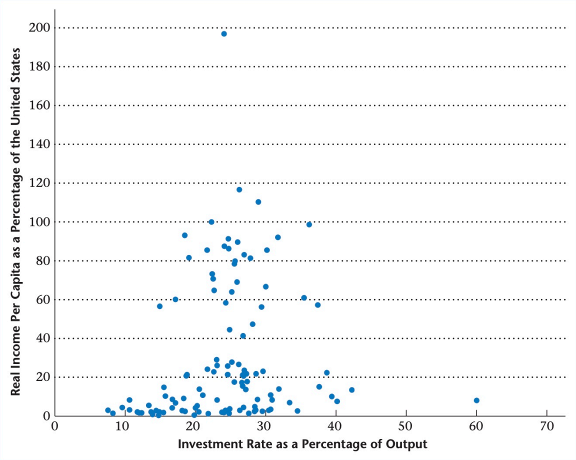
- Before Industrial Revolution, standards of living differed little over time and across countries
- Since industrial revolution, per capita income growth has been sustained in richest countries.

Rate of investment and output per worker

Figure 7.2 Real Income Per Capita vs. Investment Rate

The figure shows a positive correlation across the countries of the world, between the output per capita and the investment rate.

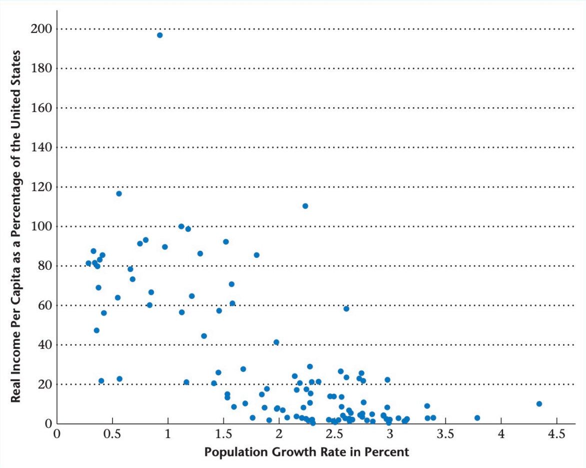
Source: A. Heston, R. Summers, and B. Aten, *Penn World Table Version 7.0*, Center for International Comparisons of Production, Income, and Prices at the University of Pennsylvania, May 2011.



Population growth rate and output per worker

Figure 7.3 Real Per Capita Income vs. the Population Growth Rate

Across the countries in the world, real per capita income and the population growth rate are negatively correlated.



Overview

1. Malthusian Model
    - Population dynamics
    - Steady state
    - steady state analysis
        - productivity increases
        - Population control
2. Solow Model: Exogenous Growth
    - Key concepts: capital accumulation
    - Set up
    - Competitive Equilibrium
    - Steady State
    - Golden rule saving rates

Reference: Williamson, Macroeconomics 5th Edition, Chapter 7

Malthusian Model

Malthusian Model

Motivation

- Why do some societies struggle to escape poverty?
- Before the Industrial Revolution, economic growth was slow, and improvements in technology didn't always lead to better living standards. Why?
- Thomas Malthus argued that population growth would outpace resource growth, leading to subsistence-level incomes.

Malthusian Model

Key concepts

- First written in essay form by Malthus in 1798
- Key predictions: improvement in technology will lead to population growth.
- Standard of livings rarely improve over time
- The theory explains growth before industrial evolution in 1800 well

Malthusian Model: Set up

- Dynamic model with many periods
- Output is determined by input of land (L), labor (N), and factor productivity (z)
- Population = labor input
- Everyone works same unit of labor (no unemployment or leisure)
- Production function is constant return to scale (CRS)

$$F(L, N) = NF(L/N, 1)$$

- Think of output as food

$$Y = C$$

- No technology to store output or convert it into saving, or investment

Malthusian Model: Set up

Population dynamics

$$N' = N + Birth - Death$$
$$N' = N + N(birthrate - deathrate)$$

Think of birth rate as fertility which is an increasing function of consumption (c) or nutrition.

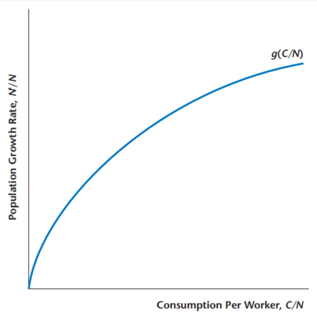
The death rate is a decreasing function of consumption, as nutrition affects infant mortality, and population health.

Population growth

Population growth is an increasing function of consumption per capita

$$\frac{N'}{N} = g(\frac{C}{N})$$

Figure 7.5 Population Growth Depends on Consumption per Worker in the Malthusian Model



There is a diminishing return in C/N to population growth

Solution

Since

$$C = Y$$
$$C = zF(L, N)$$
$$\frac{C}{N} = \frac{Z}{N}F(L, N)$$
$$= zF(\frac{L}{N}, 1)$$

From the population growth equation

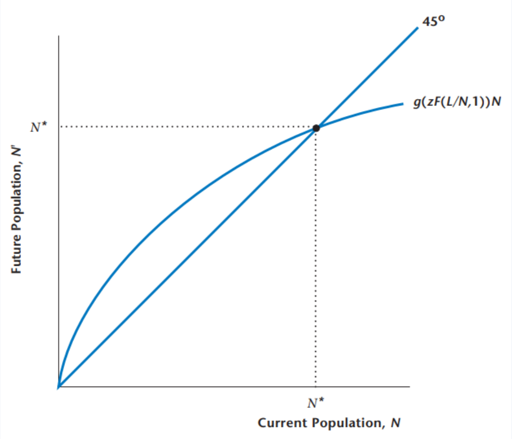
$$\frac{N'}{N} = g(\frac{C}{N})$$
$$= g(zF(\frac{L}{N}, 1))$$

Population Dynamics

There is a steady state that $N = N'$

Figure 7.6 Determination of the Population in the Steady State

In the figure, N^* is the steady state population, determined by the intersection of the curve and the 45° line. If $N > N^*$ then $N' < N$ and the population falls over time, and if $N < N^*$ then $N' > N$ and the population rises over time.



Steady State

$$N' = N = N^*$$

From figure 7.6,

- if $N > N^*, N' < N$ population decreasing
- if $N < N^*, N' > N$ population increasing

Eventually, population will converge to $N*$ in the long run equilibirum. N^* is determined by the function $F(), g()$ and value of z and L.

at N^*,

$$C^* = zF(L, N^*)$$

Per capita steady state

write output in term of per worker

$$\frac{Y}{N} = zF(\frac{L}{N}, 1)$$

per worker output

$$y = zf(l)$$

per worker consumption

$$c = zf(l)$$

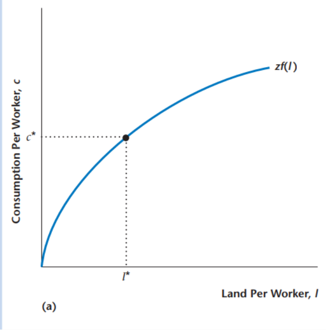
Steady state, per capita

since

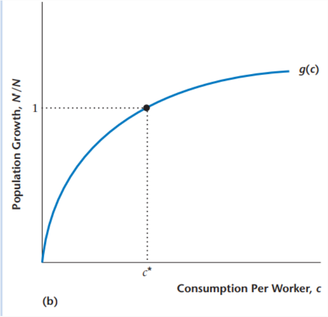
$$\frac{N'}{N} = g(c)$$

At N^*,

$$1 = g(c^*)$$
$$c^* = zf(l^*)$$



(a) steady state l



(b) steady state c

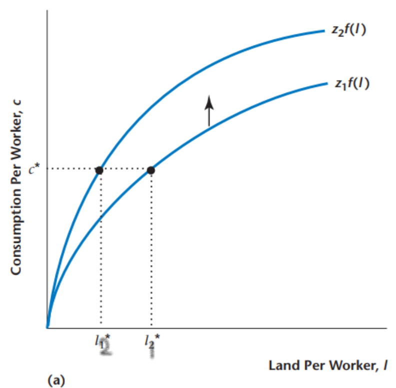
What happens when productivity rises?

when z increases, consumption per worker remains the same.

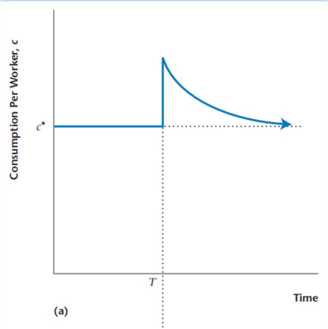
$$g(c^*) = 1$$

In the steady state, c^* is the consumption level such that birth and death rates are the same.

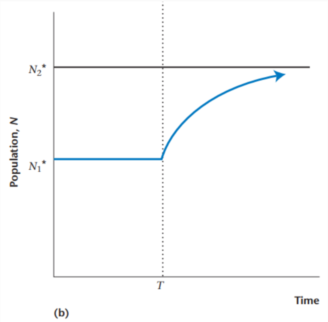
Land per worker decreases



What happens when productivity rises?



(a) Consumption per worker rises, and eventually falls down to its steady state value



(b) Population growth increases, and reaches the new equilibrium

In Mathusian model, improvement in technology do not increase population well being, as measured by consumption per capita.

Population control

How to improve welfare in Malthusian world?

Population control that only changes ”birth rate”

> *Exercise:*
> *show using graphs, how birth control affects consumption per capita, land per worker, population, and output level*

Population control

How to improve welfare in Malthusian world?

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- birth rate changes, leads to higher consumption per capita
- more land per capita
- lower population in steady state

Discussions

- Basic Mechanism
- Population as a Constraint
- Historical Evidence – Until 1800, living standards barely rose despite technological progress, supporting the Malthusian view.
- Escape from Malthusian Trap – Why did some countries break out of this cycle?
- Relevance Today
- In term of life expectancy, standard of living, population growth. What are the trends we see these days?

Solow Model: Exogenous Growth

Motivation

- How do economies grow over time, and why do some grow faster than others?
- What determines long-term prosperity?
- Why do capital investments matter but eventually show diminishing returns?

Key concepts

- Allow for saving, investment, and capital accumulation
- Make prediction about sources of growth
- More optimistic than Malthusian that we can achieve rising living standards
- Eventually diminishing return to investment
- Steady state equilibrium – eventually growth slows down
- Sustained technological growth necessary for sustained rising living standards

Consumer

$$N' = (1 + n)N$$

- n is *exogenous* population growth
- $n > -1$
- if $n < 0$, population is shrinking
- labor force is equal to population (no unemployment, or leisure)

Output

Workers are consumers. They receive all current real output (Y) as income

$$Y = C + S$$

- C: consumption
- S: saving

$$S = sY,$$

where s is saving rate.

Representative Firm

$$Y = zF(K, N)$$

- Y: current output
- z: total factor productivity
- K: current capital stock
- N: current labor input
- F(): is constant return to scale, diminishing return in inputs

Per capita notation

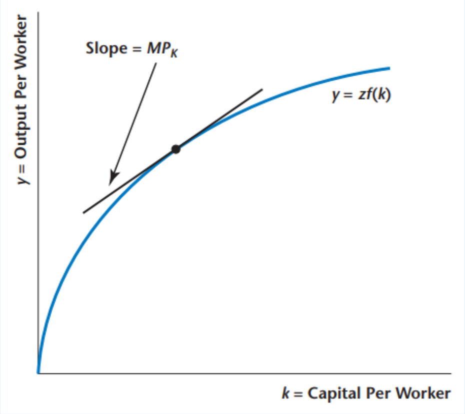
- $\frac{Y}{N} = y$ (output per worker)
- $\frac{K}{N} = k$ (capital per worker)
- $F(k, 1) = f(k)$ (per worker production function)

Production function

F(): is constant return to scale, diminishing return in inputs

Figure 7.12 The Per-Worker Production Function

This function is the relationship between aggregate output per worker and capital per worker determined by the constant-returns-to-scale production function. The slope of the per-worker production function is the marginal product of capital, MP_K.



Capital Stock

Capital accumulated over time. It depreciates each period, and increases with investment.

$$K' = (1 - d)K + I$$

- d: depreciation rate
- I: investment is from a part of output

Competitive Equilibrium

2 markets

1. Consumption (output) traded for labor
2. Consumption traded for capital

Consumers can save by accumulating capital

Labor market and capital market clear in each period

Market clearing condition
M1: labor, inelastic supply of labor = N
M2: capital clears: $S = I$, $I = sY$, *s is saving rate*

There is no government

Capital Market

From capital accumulation law of motion and market clearing conditions

$$K' = (1-d)K + I$$
$$K' = (1-d)K + sY$$
$$\frac{N'}{N'}\frac{K'}{N} = (1-d)\frac{K}{N} + sF(\frac{K}{N}, 1)$$
$$k'(1+n) = (1-d)k + sf(k)$$

$$k' = \frac{szf(k)}{1+n} + \frac{(1-d)k}{1+n}$$

Steady state

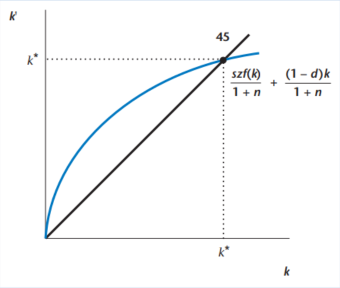
$$k' = \frac{szf(k)}{1+n} + \frac{(1-d)k}{1+n}$$

in steady state,

$$k' = k = k^*$$

Figure 7.13 Determination of the Steady State Quantity of Capital per Worker

The colored curve is the relationship between current capital per worker, k, and future capital per worker, k', determined in a competitive equilibrium in the Solow growth model. The steady state quantity of capital per worker is k^*, given by the intersection of the 45° line (the black line) with the colored curve.



Steady state

And in the long run

$$y^* = zf(k^*)$$
$$c^* = (1 - s)zf(k^*)$$

Therefore,

$$c^* \equiv k^*, s, z, n$$

There is diminishing return to investment; eventually it requires a lot of investment to increase MPK In equilibrium,

$$MPK = MRI.$$

Exogenous growth

$$k^* = \frac{K}{N}$$

Since n, population growth, is exogenous, and constant, K must grow at the same rate in equilibrium

$$I = sY = szf(k^*)N$$

.

I, investment, also grows the same rate as n

Therefore, in steady state, it requires K, I, Y, C to *grow at the same rate* as n to keep up with population growth.

Analysis of steady state

since

$$k' = \frac{szf(k)}{1+n} + \frac{(1-d)k}{1+n}$$

and

$$k' = k = k^*$$

we get

$$k^* = \frac{szf(k^*)}{1+n} + \frac{(1-d)k^*}{1+n}$$

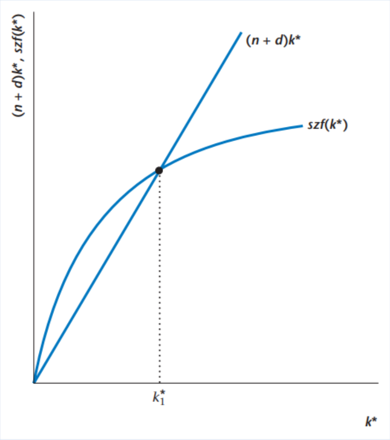
Rearrange the term to get

$$(n+d)k^* = szf(k^*)$$

Steady state capital

Figure 7.14 Determination of the Steady State Quantity of Capital per Worker

The steady state quantity of capital, k_1^* is determined by the intersection of the curve $szf(k^*)$ with the line $(n+d)k^*$.



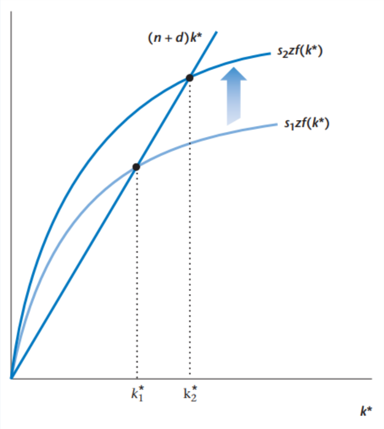
Changes in saving rates

> *Exercise*
> *What happens to k^* or c^* when there is a change in saving rates?*

Changes in saving rates

What happens to k^* or c^* when there is a change in saving rates?

Figure 7.15 Effect of an Increase in the Savings Rate on the Steady State Quantity of Capital per Worker
An increase in the savings rate shifts the curve $szf(k^*)$ up, resulting in an increase in the quantity of capital per worker from k_1^* to k_2^*.



Golden rule saving rates

Is there an optimal level of capital k^* that maximizes consumption per capita? Since

$$c^* = zf(k^*) - (n + d)k^*$$

Using first-order-condition (FOC) to find max c^*

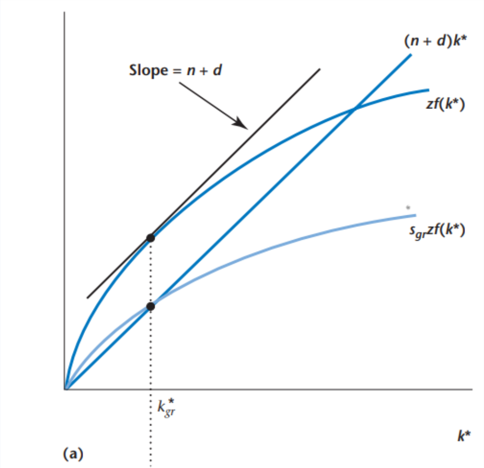
$$zf'(k^*) = n + d$$

That is k^* maximizes c^* where the slope of $zf(k^*)$ is equal to (n+d)

Golden-rule saving rate

Figure 7.18 The Golden Rule Quantity of Capital per Worker

This quantity, which maximizes consumption per worker in the steady state, is k^*_{gr}, and the maximized quantity of consumption per worker is c^{**}. The golden rule savings rate *sgr* achieves the golden rule quantity of capital per worker in a competitive equilibrium steady state.



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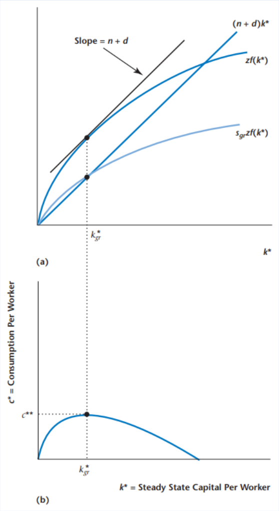

Exercise

What would happen if labor force growth rate (population growth rate) increases? ($n_2 > n_1$)