

Problem Set 1: Sets, Logics, Relations, and Functions

1. Show that $\emptyset \subset A$ for every A .
2. Prove the following statement
 - (a) $A - B = A \cap B'$
 - (b) $A \subset B$ if and only if $A - B = \emptyset$
 - (c) $A \cap (B - C) = (A \cap B) - (A \cap C)$
 - (d) $A \subset B$ if and only if $B' \subset A'$
 - (e) $A \cap A' = \emptyset$
 - (f) De Morgan Laws: $(\cup_{X \in C} X)' = \cap_{X \in C} X'$ and $(\cap_{X \in C} X)' = \cup_{X \in C} X'$
3. From the diagnostic quiz question 2, the condition for $(A \cap B) \cup C = A \cap (B \cup C)$ is that $C \subset A$. Is this a necessary and sufficient condition? Or is the equality *equivalent* to $A \subset C$.
4. Find the contrapositives of the following statement, and show whether the converses hold.
 - 1). P: n is a prime number greater than 2.
Q: n is an odd number.
Statement: $P \implies Q$.
 - 2). P: n is an even prime.
Q: $n=2$.
Statement: $P \implies Q$.
 - 3). a, b, c are integers with $a * b = c$.
P: c is odd.
Q: a, b are odd.
Statement: $P \implies Q$.
5. Consider the definition of continuity seen in class. Show graphically how this definition works. State the statement with quantifiers. Build the negation you would use to show that a function is not continuous. Show this graphically.

6. Consider Brouwer's fixed point theorem:

Suppose $X \subset \mathbb{R}$ is compact and convex, and $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous. Then there is some $x^ \in X$ such that $f(x^*) = x^*$.*

State the theorem using quantifiers. Negate it. What is the contrapositive of the statement? What is the necessary and sufficient condition of the statement?

7. When P implies Q, we say Q is necessary for P. Q implies P, Q is sufficient for P. If both, P and Q are necessary and sufficient for each other. Judge the relation between the following conditions.

1). P: The number α is divisible by two;

Q: The number α is not a prime number.

2). P: The number α is divisible by two;

Q: The number $\alpha = 4$.

3). P: The number α is divisible by two;

Q: The number α is an even number.

4). P: The number α is divisible by two;

Q: The number α is divisible by three.

8. Let $f(x) = 1/(1+x)$. Find the following, and indicate whether they are functions.

- $f(f(x))$ (for which x does this make sense?)
- $f(\frac{1}{x})$.
- $f(cx)$.
- $f(x + y)$
- $f(x) + f(y)$
- For which number c is there a number x such that $f(cx) = f(x)$. Hint: there are a lot more than you might think at first glance.

9. Quiz Q1, Q2, Q5, Q6.