

## Problem Set 2. (Weirstrass theorem, Optimization)

1. Give examples of sets and functions with the following properties
  - (a) an open set  $\mathcal{D} \subseteq \mathbb{R}$  and a continuous function  $f : \mathcal{D} \rightarrow \mathbb{R}$  that has both a maximum and a minimum on  $\mathcal{D}$
  - (b) a compact set  $\mathcal{D} \subseteq \mathbb{R}$  and two functions,  $g, h : \mathcal{D} \rightarrow \mathbb{R}$  such that neither  $g$  nor  $h$  has a maximum or minimum on  $\mathcal{D}$ , but the sum  $g + h$  does.
2. Prove the following two propositions:
  - (a)  $x^* = \arg \max_{x \in \mathcal{D}} f(x) \Leftrightarrow x^* = \arg \min_{x \in \mathcal{D}} -f(x)$
  - (b) Let  $f : \mathcal{D} \rightarrow \mathbb{R}$  and  $g : \mathbb{R} \rightarrow \mathbb{R}$ , with  $g' > 0$ . Then  $x \in \mathcal{D}$  is a maximum on  $f$  if and only if  $x$  is a maximum on  $g(f(x))$ .
3. Let  $\mathcal{D} = \{0\} \cup \{\frac{1}{n}, n = 1, 2, \dots\}$ . Determine if the following statements are true.
  - (a) If  $f : \mathcal{D} \rightarrow \mathbb{R}$  is continuous, then  $f$  has a maximum on  $\mathcal{D}$ .
  - (b) If  $g : \mathcal{D} \rightarrow \mathbb{R}$  is a function so that  $g(x) \in [0, 1]$ , then  $g$  has a maximum or a minimum on  $\mathcal{D}$ .
4. Solve exercises 1, 2, 4, 9, 14 and 15 in Sundaram, Chapter 3.